Noetherian Property of Invariant Rings

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Overview

Noetherian Ring

Three equivalent definitions of a Noetherian ring:

- All ideals of the ring are finitely generated
- satisfies the ascending chain condition: all ascending chains of ideals must stabilize.
- every nonempty subset of ideals has a maximal element

Examples

Let k be a field. Then it only has two ideals: the zero ideal and itself. Therefore k is Noetherian.

Examples

$k[x_1, x_2, \cdots, x_n]$ is Noetherian where k is a field.

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$k[x_1, x_2, \cdots]$ is not Noetherian because we can take the chain $(x_1) \subsetneq (x_1, x_2) \subsetneq \cdots$

Group actions

Let S be a set and G be a group. A (left) group action of G on S is a map

$$G \times S \to S$$

$$(g, s) \mapsto gs$$

such that $1_G s = s$ and $(gh)s = g(hs)$ for any $g, h \in G$ and any $s \in S$.

Invariant rings

If we have a group G acting on a ring R by ring automorphisms which means that there is a group homomorphism

$$G \to AutR$$

then the invariant ring is

$$R^G := \{r \in R \mid gr = r \text{ for every } g \in G\}$$

Theorem

Let G be a finite group and let R be a Noetherian ring which contains $\mathbb{Q}$. Then if G acts on R, the invariant ring R^G is also Noetherian.

Reynold's Operator

Let $\rho : R \to R^G$ be the map

$$\rho(r) := \frac{1}{|G|} \sum_{g \in G} g(r)$$

Emmy Noether's Theorem 1926

Let $R = k[x_1, \cdots, x_n]$ where k is a field.
G is a finite subgroup of $GL(n, k)$ Then R^G is Noetherian.

GL(n,k)

$GL(n, k)$ elements are $n \times n$ invertible matrices and act via matrix multiplication on elements of R:

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \cdots & \cdots & \cdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ \cdots \\ x_n \end{pmatrix}$$

Extension of Nagarajan's example for positive characteristic p

Let p be a prime integer.

$$k := \mathbb{F}_p(a_1, b_1, a_2, b_2, \cdots)$$

$$R := k[\![x, y]\!]$$

We define an automorphism $\sigma : R \to R$ where

- $\sigma(x) = x$
- $\sigma(y) = y$
- $\sigma(a_n) = a_n + yP_{n+1}$
- $\sigma(b_n) = b_n - xP_{n+1}$

$$P_n := a_n x - b_n y$$

Then R^G is not Noetherian under the group action of $\langle \sigma \rangle$.

Properties of the automorphism

σ generates a finite group from these properties:

- $\sigma(P_n) = P_n$
- $\sigma^p(a_n) = a_n$
- $\sigma^p(b_n) = b_n$

Thus

$$\langle \sigma \rangle \cong \mathbb{Z}/p\mathbb{Z}$$

References

- Nagarajan, K. R. (1968)
- Nagata, Masayoshi (1969)
- Emmy Noether (1926)

Thank you!